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1986 J. Phys. A: Math. Gen. 19 1783

(http://iopscience.iop.org/0305-4470/19/10/019)

J. Phys. A: Math. Gen. 19 (1986) 1783-1791. Printed in Great Britain

Painlevé analysis and integrability of coupled non-linear Schrödinger equations

R Sahadevan, K M Tamizhmani and M Lakshmanan

Department of Physics, Bharathidasan University, Tiruchirapalli 620 023, India

Received 9 May 1985

Abstract. Considering a system of coupled non-linear Schrödinger (NLS) equations, we discuss the integrability properties through Painlevé (P) analysis. For the two coupled NLS equations, we show that there exists a pair of parametric values possessing the P property, for which the associated Bäcklund transformation (BT) and the Hirota bilinearisation are constructed. These parametric choices are identical to those of Zakharov and Schulman who established the integrability in terms of 'motion invariants'. Finally, we extend the P analysis to the N coupled NLS system, and identify two parametric choices possessing the P property, which are natural generalisations of the two coupled NLS cases.

1. Introduction

In recent years the singular point structural analysis, leading to the Painlevé (P) property advocated originally by Ablowitz *et al* (1980) for non-linear evolution equations, has played a key role in identifying integrable non-linear dynamical systems (Chang *et al* 1982, Ramani *et al* 1982, Tabor and Weiss 1981, Lakshmanan and Sahadevan 1985). The objective of the P analysis is to locate the presence of movable critical points (algebraic and logarithmic branch points and essential singularities) exhibited by the general solution in the complex time plane, and prove that the solution is meromorphic (or transformable to meromorphic). The concept of P property has been appropriately generalised by Weiss *et al* (1983) to test the integrability behaviour of non-linear partial differential equations (NPDE). The modified definition is that a PDE has the P property if its general solution is single-valued about the non-characteristic movable singularity manifold. In other words, if the singularity manifold is determined by

$$\phi(x, t) = 0 \qquad \phi_x(x, t) \neq 0 \tag{1}$$

and $u = u(x, t)$ is a solution of the PDE, we require that

$$u = \phi^{\alpha} \sum_{j=0}^{\infty} u_j \phi^j, \tag{2}$$

where $u_0 \neq 0$, $\phi = \phi(x, t)$, $u_j = u_j(x, t)$ are analytic functions of (x, t) in a neighbourhood of the manifold (1) and α is a negative integer. The remarkable feature of the P test, particularly for soliton equations, is that a natural connection exists in relation to the Lax pairs, BT, integrability, etc, and (2). This Painlevé test has been successfully performed for a class of non-linear evolution equations in order to show their integrability (Weiss 1983, 1984a, b, Steeb *et al* 1984, Gibbon and Tabor 1985, Gibbon *et al* 1985).

0305-4470/86/101783+09$02.50

In this paper, we consider a system of two coupled NLS equations defined by

$$\mathrm{i}\chi_{1t} = c_1\chi_{1xx} + 2\alpha|\chi_1|^2\chi_1 + 2\beta|\chi_2|^2\chi_1 \tag{3a}$$

$$\mathrm{i}\chi_{2t} = c_2\chi_{2xx} + 2\gamma|\chi_2|^2\chi_2 + 2\beta|\chi_1|^2\chi_2 \tag{3b}$$

and more generally a system of N coupled equations

$$\mathrm{i}\chi_{kt} = c_k\chi_{kxx} + 2\alpha_{kk}|\chi_k|^2\chi_k + 2\sum_{\substack{l=1\\(l\neq k)}}^{N} \alpha_{kl}|\chi_l|^2\chi_k,$$

$$\alpha_{kl} = \alpha_{lk}, \qquad k,\ l = 1, 2, \ldots, N, \tag{4}$$

where c_1, c_2, c_k, α, β, γ and α_{kl} are parametric constants and subscripts denote partial differentiation. For the system (3), Zakharov and Schulman (1982) established the complete integrability in terms of 'motion invariants' by using the degenerative dispersion laws for the specific parametric restrictions

$$\alpha = \beta = \gamma \qquad c_1 = c_2 \tag{5a}$$

$$\alpha = -\beta = \gamma \qquad c_1 = -c_2. \tag{5b}$$

Furthermore, this result can also be proved by deriving the appropriate IST formalism (Zakharov and Schulman 1982 and references therein). In this paper through a systematic search of P properties (§ 2) of the system (3) we establish that only for the parametric choices given by (5) is the system free from movable critical singularity manifolds and thereby integrable. We also derive the corresponding Bäcklund transformation and construct the Hirota (1974a, b) bilinearisation (§ 3). As a consequence, the construction of an N-soliton solution is pointed out. Finally, considering the N coupled system (4), we show that the P property holds (§ 4) for a certain generalisation of (5) for which a suitable linear eigenvalue problem is also known to exist (Zakharov and Manakov 1975).

2. P analysis of two coupled NLS equations

In order to investigate the integrability properties of the system (3), we rewrite it in terms of four real functions P, Q, R and S defined by $\chi_1 = P + \mathrm{i}Q$ and $\chi_2 = R + \mathrm{i}S$. Consequently, we have the following equations:

$$-Q_t = c_1 P_{xx} + 2\alpha(P^2 + Q^2)P + 2\beta(R^2 + S^2)P \tag{6a}$$

$$P_t = c_1 Q_{xx} + 2\alpha(P^2 + Q^2)Q + 2\beta(R^2 + S^2)Q \tag{6b}$$

$$-S_t = c_2 R_{xx} + 2\beta(P^2 + Q^2)R + 2\gamma(R^2 + S^2)R \tag{6c}$$

$$R_t = c_2 S_{xx} + 2\beta(P^2 + Q^2)S + 2\gamma(R^2 + S^2)S. \tag{6d}$$

As usual (Weiss *et al* 1983), the P analysis consists essentially of three stages: (i) determination of the leading-order behaviour, (ii) identifying the resonances and (iii) verifying that a sufficient number of arbitrary functions exists without the introduction of movable critical singularity manifolds. To start with let us assume that the leading orders are of the form

$$P \approx P_0\phi^{\alpha_1} \qquad Q \approx Q_0\phi^{\alpha_2} \qquad R \approx R_0\phi^{\alpha_3} \qquad S \approx S_0\phi^{\alpha_4} \tag{7}$$

where α_1, α_2, α_3 and α_4 are integers to be determined and $\phi(x, t)$ is the singularity manifold. By using (7) in (6) and equating the most dominant terms we obtain the unique choice

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -1 \tag{8}$$

and

$$\alpha(P_0^2 + Q_0^2) + \beta(R_0^2 + S_0^2) = -c_1\phi_x^2 \tag{9a}$$

$$\beta(P_0^2 + Q_0^2) + \gamma(R_0^2 + S_0^2) = -c_2\phi_x^2. \tag{9b}$$

We note that the remaining equations resulting from (6) are identical to the set (9). Therefore, at this stage we conclude that two of the four functions P_0, Q_0, R_0 and S_0 are arbitrary without any parametric constraints.

2.1. Resonances

For finding the powers at which the arbitrary functions can enter into the series, we substitute the expressions

$$\begin{aligned} P &\approx P_0\phi^{-1} + P_j\phi^{j-1} \qquad & Q &\approx Q_0\phi^{-1} + Q_j\phi^{j-1} \\ R &\approx R_0\phi^{-1} + R_j\phi^{j-1} \qquad & S &\approx S_0\phi^{-1} + S_j\phi^{j-1} \end{aligned} \tag{10}$$

into (6), and comparing the lowest-order terms we obtain a system of four linear algebraic equations in (P_j, Q_j, R_j, S_j). In matrix form it may be conveniently written as

$$[A_2(j)][\Omega] = 0 \qquad [\Omega] = (P_j, Q_j, R_j, S_j)^{\mathrm{T}}. \tag{11}$$

To have a non-trivial solution for (P_j, Q_j, R_j, S_j) we demand that

$$\det A_2(j) = \begin{vmatrix} c_1 j' + 4\alpha P_0^2 & 4\alpha P_0 Q_0 & 4\beta P_0 R_0 & 4\beta P_0 S_0 \\ 4\alpha P_0 Q_0 & c_1 j' + 4\alpha Q_0^2 & 4\beta Q_0 R_0 & 4\beta Q_0 S_0 \\ 4\beta P_0 R_0 & 4\beta Q_0 R_0 & c_2 j' + 4\gamma R_0^2 & 4\gamma R_0 S_0 \\ 4\beta P_0 S_0 & 4\beta Q_0 S_0 & 4\gamma R_0 S_0 & c_2 j' + 4\gamma S_0^2 \end{vmatrix} = 0 \tag{12}$$

where

$$j' = (j^2 - 3j)\phi_x^2.$$

From a knowledge of the basic properties of the determinants we can easily deduce that

$$\det A_2(j) = j^2(j-3)^2(j^2 - 3j + 4)\left(j^2 - 3j + \frac{4}{c_1 c_2 \phi_x^2}(R_0^2 + S_0^2)(c_1\gamma - c_2\beta)\right) = 0 \tag{13a}$$

and so the resonance values are

$$j = -1, 0, 0, 3, 3, 4, \frac{3}{2} \pm \frac{1}{2}\left(9 - \frac{16}{c_1 c_2 \phi_x^2}(R_0^2 + S_0^2)(c_1\gamma - c_2\beta)\right)^{1/2}. \tag{13b}$$

Obviously the resonance at $j = -1$ corresponds to the arbitrariness of $\phi(x, t)$. Furthermore, for Painlevé the resonances must be non-negative integers. This requirement leads to the following two possibilities.

Case 1

$$c_1 c_2 \phi_x^2 = 2(R_0^2 + S_0^2)(c_1\gamma - c_2\beta) \tag{14a}$$

and the associated resonances are

$$j = -1, 0, 0, 1, 2, 3, 3, 4. \tag{14b}$$

Case 2

$$(R_0^2+S_0^2)(c_1\gamma-c_2\beta)=0 \tag{15a}$$

and so (13*b*) reduces to

$$j=-1, 0, 0, 0, 3, 3, 3, 4. \tag{15b}$$

Thus we have isolated two sets of resonances given in (14) and (15) and the parametric constants obey (9).

2.2. *Arbitrary functions*

For computing the arbitrary functions at the resonance values we now introduce the following series expansions,

$$\begin{aligned} P &\approx P_0\phi^{-1}+\sum_{j=1}^{4} P_j\phi^{j-1} \qquad & Q &\approx Q_0\phi^{-1}+\sum_{j=1}^{4} Q_j\phi^{j-1} \\ R &\approx R_0\phi^{-1}+\sum_{j=1}^{4} R_j\phi^{j-1} \qquad & S &\approx S_0\phi^{-1}+\sum_{j=1}^{4} S_j\phi^{j-1} \end{aligned} \tag{16}$$

into the full equations (6). We shall discuss the evaluation of arbitrary functions of the cases 1 and 2 separately.

2.2.1. Case 1. From equation (14), we have the resonances at $j=-1$, 0, 0, 1, 2, 3, 3, 4. It is obvious that the arbitrary functions P_0, Q_0 (say), pointed out in the earlier discussion correspond to the resonance values 0, 0. However, further calculations show that none of the functions P_1, Q_1, R_1 or S_1 is arbitrary. Hence we conclude that the associated series solution will have a lesser number of arbitrary functions only and so the resulting solution does not correspond to the general solution.

2.2.2. Case 2. Here the resonance values are $j=-1, 0, 0, 0, 3, 3, 3, 4$ and the parameters satisfy the condition $(R_0^2+S_0^2)(c_1\gamma-c_2\beta)=0$. From the leading-order analysis we have

$$\alpha(P_0^2+Q_0^2)+\beta(R_0^2+S_0^2)=-c_1\phi_x^2 \qquad \beta(P_0^2+Q_0^2)+\gamma(R_0^2+S_0^2)=-c_2\phi_x^2$$

where the functions P_0 and Q_0 are arbitrary. Furthermore, for the P property we demand that the function R_0 (or S_0) be arbitrary in addition to P_0 and Q_0. This is possible only for the following two parametric restrictions,

$$\alpha=\beta=\gamma \qquad c_1=c_2 \tag{17a}$$

$$\alpha=-\beta=\gamma \qquad c_1=-c_2 \tag{17b}$$

which are identical to (5) obtained by Zakharov and Schulman (1982).

Proceeding further with the choice (17*a*), and equating the coefficients of $(\phi^{-2}, \phi^{-2}, \phi^{-2}, \phi^{-2})$ to zero, when (16) is used in (6), we obtain (after simplification)

$$\begin{bmatrix} 2(2\alpha P_0^2-c_1\phi_x^2) & 4\alpha P_0Q_0 & 4\alpha P_0R_0 & 4\alpha P_0S_0 \\ 4\alpha P_0Q_0 & 2(2\alpha Q_0^2-c_1\phi_x^2) & 4\alpha Q_0R_0 & 4\alpha Q_0S_0 \\ 4\alpha P_0R_0 & 4\alpha Q_0R_0 & 2(2\alpha R_0^2-c_1\phi_x^2) & 4\alpha R_0S_0 \\ 4\alpha P_0S_0 & 4\alpha Q_0S_0 & 4\alpha R_0S_0 & 2(2\alpha S_0^2-c_1\phi_x^2) \end{bmatrix} \begin{bmatrix} P_1 \\ Q_1 \\ R_1 \\ S_1 \end{bmatrix} = [X] \tag{18a}$$

where the column matrix $[X]$ is given by

$$[X]=\begin{bmatrix} Q_0\phi_t+2c_1P_{0x}\phi_x+c_1P_0\phi_{xx} \\ -P_0\phi_t+2c_1Q_{0x}\phi_x+c_1Q_0\phi_{xx} \\ S_0\phi_t+2c_1R_{0x}\phi_x+c_1R_0\phi_{xx} \\ -R_0\phi_t+2c_1S_{0x}\phi_x+c_1S_0\phi_{xx} \end{bmatrix}. \tag{18b}$$

By solving (18) for a unique solution, we find, after completing the next two stages, that

$$P_1=Q_1=R_1=S_1=0 \qquad [X]=[0]. \tag{19}$$

Similarly by comparing the coefficients $(\phi^{-1},\phi^{-1},\phi^{-1},\phi^{-1})$ in (6), we see that the functions P_2, Q_2, R_2 and S_2 vanish and P_0, Q_0, R_0 and S_0 satisfy

$$Q_{0t}+c_1P_{0xx}=0 \tag{20a}$$

$$P_{0t}-c_1Q_{0xx}=0 \tag{20b}$$

$$S_{0t}+c_1R_{0xx}=0 \tag{20c}$$

$$R_{0t}-c_1S_{0xx}=0. \tag{20d}$$

On the other hand the coefficients of $(\phi^0,\phi^0,\phi^0,\phi^0)$ in (6) reduce to a single equation

$$P_0P_3+Q_0Q_3+R_0R_3+S_0S_3=0 \tag{21}$$

so that three of the four functions P_3, Q_3, R_3 and S_3 are arbitrary. In a similar way we easily verify that any one of the functions P_4, Q_4, R_4 and S_4 is arbitrary, provided

$$Q_3\phi_t+2c_1P_{3x}\phi_x+c_1P_3\phi_{xx}=0 \tag{22a}$$

$$P_3\phi_t-2c_1Q_{3x}\phi_x-c_1Q_3\phi_{xx}=0 \tag{22b}$$

$$S_3\phi_t+2c_1R_{3x}\phi_x+c_1R_3\phi_{xx}=0 \tag{22c}$$

$$R_3\phi_t-2c_1S_{3x}\phi_x-c_1S_3\phi_{xx}=0 \tag{22d}$$

hold. Thus the P property is satisfied for the parametric restriction (17*a*).

In an altogether analogous way we check that the two coupled NLS equations (3) possess the P property for the choice (17*b*) also, but we refrain from giving the details here. Finally, we verified that for each one of the parametric choices (17*a*) and (17*b*), the other solution branch associated with the resonance (14*b*) (case 1) admits only a lesser number of arbitrary functions and is free from movable critical singularity manifolds. Thus equations (3) indeed possess the P property only for the Zakharov–Schulman choices (5).

3. Bäcklund transformations and the Hirota bilinearisation

In order to derive the BT of the above two P cases, we truncate the series (16) up to a constant level term, that is, P_j, Q_j, R_j and S_j are equal to zero for $j\geqslant 2$. Thus the associated BT leads to

$$\begin{aligned} P&=P_0\phi^{-1}+P_1 \qquad & Q&=Q_0\phi^{-1}+Q_1 \\ R&=R_0\phi^{-1}+R_1 \qquad & S&=S_0\phi^{-1}+S_1 \end{aligned} \tag{23}$$

where P_1, Q_1, R_1 and S_1 satisfy the equations (6). We check then that if P_0, Q_0, R_0 and S_0 satisfy (18) and (20), then P, Q, R and S are indeed solutions to (6). Considering the special case $P_1 = Q_1 = R_1 = S_1 = 0$, and using (23) in (6), we obtain the following equations for the parametric choice (17*a*):

$$[2c_1(\phi_x^2 - \phi\phi_{xx}) + 2\alpha\Gamma]P_0 = [Q_{0t}\phi - Q_0\phi_t + c_1(P_{0xx}\phi + P_0\phi_{xx} - 2P_{0x}\phi_x)]\phi \tag{24a}$$

$$[2c_1(\phi_x^2 - \phi\phi_{xx}) + 2\alpha\Gamma]Q_0 = [P_0\phi_t - P_{0t}\phi + c_1(Q_{0xx}\phi + Q_0\phi_{xx} - 2Q_{0x}\phi_x)]\phi \tag{24b}$$

$$[2c_1(\phi_x^2 - \phi\phi_{xx}) + 2\alpha\Gamma]R_0 = [S_{0t}\phi - S_0\phi_t + c_1(R_{0xx}\phi + R_0\phi_{xx} - 2R_{0x}\phi_x)]\phi \tag{24c}$$

$$[2c_1(\phi_x^2 - \phi\phi_{xx}) + 2\alpha\Gamma]S_0 = [R_0\phi_t - R_{0t}\phi + c_1(S_{0xx}\phi + S_0\phi_{xx} - 2S_{0x}\phi_x)]\phi \tag{24d}$$

where

$$\Gamma = P_0^2 + Q_0^2 + R_0^2 + S_0^2.$$

Equation (24) can be rewritten in terms of Hirota's bilinear operators (Hirota 1974a, b) in the following way:

$$(-c_1 D_x^2\phi\cdot\phi + 2\alpha\Gamma)P_0 + (D_t Q_0\phi + c_1 D_x^2 P_0\phi)\phi = 0 \tag{25a}$$

$$(c_1 D_x^2\phi\cdot\phi - 2\alpha\Gamma)Q_0 + (D_t P_0\phi - c_1 D_x^2 Q_0\phi)\phi = 0 \tag{25b}$$

$$(-c_1 D_x^2\phi\cdot\phi + 2\alpha\Gamma)R_0 + (D_t S_0\phi + c_1 D_x^2 R_0\phi)\phi = 0 \tag{25c}$$

$$(c_1 D_x^2\phi\cdot\phi - 2\alpha\Gamma)S_0 + (D_t R_0\phi - c_1 D_x^2 S_0\phi)\phi = 0 \tag{25d}$$

where the D operators are defined by

$$D_t^n D_x^m f\cdot g = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m f(x,t)g(x,t) \qquad \text{at } x' = x,\ t' = t.$$

Equations (25) can also be decoupled as

$$(c_1 D_x^2 - \mu)\phi\cdot\phi = 2\alpha\Gamma \tag{26a}$$

$$D_t Q_0\phi + c_1 D_x^2 P_0\phi - \mu P_0\phi = 0 \tag{26b}$$

$$-D_t P_0\phi + c_1 D_x^2 Q_0\phi - \mu Q_0\phi = 0 \tag{26c}$$

$$D_t S_0\phi + c_1 D_x^2 R_0\phi - \mu R_0\phi = 0 \tag{26d}$$

$$-D_t R_0\phi + c_1 D_x^2 S_0\phi - \mu S_0\phi = 0 \tag{26e}$$

where μ is a constant to be determined. In particular, from equation (26*a*), we have

$$P^2 + Q^2 + R^2 + S^2 = \frac{1}{\alpha}\left(c_1 \frac{\partial^2}{\partial x^2}\log\phi - \frac{\mu}{2}\right). \tag{27}$$

Now expanding the functions ϕ, P_0, Q_0, R_0 and S_0 as power series (Hirota 1974a, b) and using them in (27), we can construct the N soliton solutions in the usual way. Furthermore, we have checked that the above Hirota bilinearisation holds good for the parametric choice (17*b*) also, except for the fact that (27) now has to be replaced by

$$P^2 + Q^2 - (R^2 + S^2) = \frac{1}{\alpha}\left(c_1 \frac{\partial^2}{\partial x^2}\log\phi - \frac{\mu}{2}\right).$$

4. P analysis of the N coupled NLS system

In the previous sections, we discussed the P analysis of two coupled NLS equations and identified two specific P cases. In this section we extend the P analysis to the N coupled NLS system given in (4), whose Hamiltonian is of the form

$$H=\int_{-\infty}^{\infty}\left(\sum_{k=1}^{N} c_k|\chi_{kx}|^2+\sum_{k=1}^{N}\alpha_{kk}|\chi_k|^4+\sum_{\substack{k,l=1\\(k\neq l)}}^{N}\alpha_{kl}|\chi_k|^2|\chi_l|^2\right)\mathrm{d}x. \tag{28}$$

For application of the P analysis to the system (4), we rewrite it as

$$-Q_{kt}=c_kP_{kxx}+2\alpha_{kk}(P_k^2+Q_k^2)P_k+2\sum_{\substack{l=1\\(l\neq k)}}^{N}\alpha_{kl}(P_l^2+Q_l^2)P_k \tag{29a}$$

$$P_{kt}=c_kQ_{kxx}+2\alpha_{kk}(P_k^2+Q_k^2)Q_k+2\sum_{\substack{l=1\\(l\neq k)}}^{N}\alpha_{kl}(P_l^2+Q_l^2)Q_k \tag{29b}$$

where

$$\chi_k=P_k+\mathrm{i}Q_k \qquad k=1,2,\ldots,N.$$

Now we assume that the leading orders are of the form

$$P_k\approx P_{k0}\phi^{\beta_{1k}} \qquad Q_k\approx Q_{k0}\phi^{\beta_{2k}} \qquad k=1,2,\ldots,N \tag{30}$$

where β_{1k} and β_{2k} are integers to be determined. For this, we substitute (30) into (29) and equate the most singular terms to obtain

$$\beta_{1k}=\beta_{2k}=-1 \tag{31a}$$

and

$$\alpha_{kk}(P_{k0}^2+Q_{k0}^2)+\sum_{\substack{l=1\\(l\neq k)}}^{N}\alpha_{kl}(P_{l0}^2+Q_{l0}^2)=-c_k\phi_x^2 \qquad k=1,2,\ldots,N. \tag{31b}$$

Here also we note that the $2N$ equations resulting from (29a) and (29b) reduce to N equations (31b). Thus we infer that N functions out of the $2N$ functions of P_{k0} and Q_{k0} are arbitrary.

As before, carrying out the resonance analysis (see, for example, Lakshmanan and Sahadevan 1985), we obtain $4N$ resonance values and, computing the associated arbitrary functions, we find that the series type solution is free from movable critical singularity manifolds only for the following specific parametric restrictions:

$$\alpha_{kk}=\alpha_{kl} \qquad c_1=c_m \tag{32a}$$

$$\alpha_{kk}=-\alpha_{kl} \qquad c_1=-c_m \qquad k,l,m=1,2,\ldots,N,\ (k\neq l). \tag{32b}$$

We shall briefly outline the details below for the choice (32a), while the procedure is analogous for (32b).

It is easy to check that the associated resonance values can be written as

$$j=-1,0,0,\ldots,(2N-1)\text{ times},\ 3,3,\ldots,(2N-1)\text{ times},\ 4. \tag{33a}$$

To evaluate the required $(4N-1)$ arbitrary functions, we substitute

$$P_k \approx P_{k0}\phi^{-1} + \sum_{\mu=1}^{4} P_{k\mu}\phi^{\mu-1}$$
$$Q_k \approx Q_{k0}\phi^{-1} + \sum_{\mu=1}^{4} Q_{k\mu}\phi^{\mu-1} \qquad k=1,2,\ldots,N \tag{33b}$$

in the full equations (29). Obviously the parametric constraint (32*a*) further reduces the N equations in (31*b*) into a single equation and so $(2N-1)$ functions of P_{k0} and Q_{k0}, $k=1,2,\ldots,N$, become arbitrary. Now by equating the coefficients of $(\phi^{-2}, \phi^{-2}, \ldots, \phi^{-2})$ and $(\phi^{-1}, \phi^{-1}, \ldots, \phi^{-1})$ to zero in (29), we find that $P_{k\mu}$, $Q_{k\mu}$, $k=1,2,\ldots,N$; $\mu=1$, 2 vanish, provided

$$Q_{k0}\phi_t + 2c_1 P_{k0}\phi_x + c_1 P_{k0}\phi_{xx} = 0 \tag{34a}$$
$$P_{k0}\phi_t + 2c_1 Q_{k0}\phi_x - c_1 Q_{k0}\phi_{xx} = 0 \tag{34b}$$

and

$$Q_{k0t} + c_1 P_{k0xx} = 0 \tag{35a}$$
$$P_{k0t} - c_1 Q_{k0xx} = 0 \qquad k=1,2,\ldots,N \tag{35b}$$

hold respectively. Furthermore, the comparison of the coefficients $(\phi^0, \phi^0, \ldots, \phi^0)$ in $2N$ equations of (29) gives rise to a single equation

$$\sum_{k=1}^{N} (P_{k0}P_{k3} + Q_{k0}Q_{k3}) = 0 \tag{36}$$

and hence $(2N-1)$ functions of P_{k3} and Q_{k3}, $k=1,2,\ldots,N$ are arbitrary. In a similar manner we can easily verify that either one of the functions of P_{k4} or Q_{k4} is arbitrary, while the remaining functions are expressible in terms of the others if ϕ, P_{k3} and Q_{k3} satisfy the following:

$$Q_{k3}\phi_t + 2c_1 P_{k3x}\phi_x + c_1 P_{k3}\phi_{xx} = 0 \tag{37a}$$
$$P_{k3}\phi_t - 2c_1 Q_{k3x}\phi_x - c_1 Q_{k3}\phi_{xx} = 0. \tag{37b}$$

Again we checked that for each one of the parametric choices (32*a*) and (32*b*), the remaining solution branch possesses only a lesser number of arbitrary functions and also verified that it does not introduce any movable critical singularity manifolds as in the case of the two coupled NLS equations system. Thus the N coupled NLS equations (29) possess the P property for the parametric choices (32*a*) and (32*b*) and so are integrable. It is also interesting to note that complete integrability has also been proved for the above parametric choices (32*a*) and (32*b*) through the knowledge of linear eigenvalue problems (Zakharov and Manakov 1975) and also by deriving an infinite number of commutative Lie-Bäcklund transformations (Zhiber 1982).

5. Discussion

In this paper, we systematically analysed the P properties of a two coupled NLS equations system and showed that there exist two Painlevé cases, which are integrable. Also, the associated BT and their connection with Hirota bilinear formalism was explored, so that N soliton solutions could be constructed. Finally, we applied the P analysis to the N coupled NLS equations system and identified two P cases, which are the natural generalisations of two coupled NLS equation cases.

Acknowledgments

The work reported here forms part of an Indian National Science Academy Research Project. One of us (KMT) acknowledges the Department of Atomic Energy, Government of India for providing a Senior Research Fellowship.

References

Ablowitz M J, Ramani A and Segur H 1980 *J. Math. Phys.* **21** 715
Chang Y F, Tabor M and Weiss J 1982 *J. Math. Phys.* **23** 531
Gibbon J D, Radmore P, Tabor M and Wood P 1985 *Stud. Appl. Math.* **72** 39
Gibbon J D and Tabor M 1985 *Dynamical Problems in Soliton Systems* ed S Takeno (Berlin: Springer)
Hirota R 1974a *Bäcklund Transformations, the Inverse Scattering Methods, Solitons and their Applications* ed R M Miura (Berlin: Springer)
—— 1974b *Prog. Theor. Phys.* **52** 1498
Lakshmanan M and Sahadevan R 1985 *Phys. Rev.* A **31** 861
Ramani A, Dorizzi B and Grammaticos B 1982 *Phys. Rev. Lett.* **49** 1539
Steeb W-H, Kloke M and Spiker B-M 1984 *J. Phys. A: Math. Gen.* **17** L825
Tabor M and Weiss J 1981 *Phys. Rev.* A **24** 2157
Weiss J 1983 *J. Math. Phys.* **24** 1405
—— 1984a *J. Math. Phys.* **25** 13
—— 1984b *J. Math. Phys.* **25** 2226
Weiss J, Tabor M and Carnevale G 1983 *J. Math. Phys.* **24** 522
Zakharov V E and Manakov S V 1975 *Zh. Eksp. Teor. Fiz.* **69** 1651
Zakharov V E and Schulman E I 1982 *Physica* **4D** 270
Zhiber A V 1982 *Teor. Mat. Fiz.* **52** 405